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Relative Extinction of Heterogeneous Agents

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Relative Extinction of Heterogeneous Agents*

Jaksa Cvitanic and Semyon Malamud

Abstract

In all the existing literature on survival in heterogeneous economies, the rate at which an agent vanishes in the long run relative to another agent can be characterized by the difference of the so-called survival indices, where each survival index only depends on the preferences of the corresponding agent and the properties of the aggregate endowment. In particular, one agent experiences extinction relative to another (that is, the wealth ratio of the two agents goes to zero) if and only if she has a smaller survival index. We consider a simple complete market model and show that the survival index is more complex if there are more than two agents in the economy. In fact, the following phenomenon may take place: even if agent one experiences extinction relative to agent two, adding a third agent to the economy may reverse the situation and force the agent two to experience extinction relative to agent one. We also calculate the rates of convergence.

KEYWORDS: equilibrium, heterogeneous agents, survival, extinction

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1 Introduction

Long run survival of economic agents in competitive environments has been extensively studied in many areas of economics, such as financial economics (see e.g., Blume and Easley (1992, 2002, 2006, 2009a, 2009b)), firm competition (see e.g., Sjöström and Weitzman (1996)) and evolutionary games (see e.g., Samuelson (1997)). As much as survival is important for understanding the long run equilibrium behavior, it is equally important to understand how quickly the non-surviving agents get extinct. If the rate of extinction is low, they will impact equilibrium behavior for a long time. There is then a need to study what happens during this time, before the agents get extinct. Do they get extinct simultaneously or gradually, one after another? For example, if the extinction of agent i relative to agent j is fast, agent j will still own a large part of the market for a long time until he gets extinct relative to the surviving agents. This raises the following important question: if we want to know whether agent i dominates over j in the long run, can we consider them in isolation? Is knowing the characteristics of the two agents sufficient for answering the extinction question? The main goal of this paper is to provide a simple example illustrating that, generally, the answer to this question is "No." In fact, we show that an "extinction reversal" may occur: even if agent i experiences extinction relative to agent j, adding a new agent to the economy may reverse the situation and force agent j to experience extinction relative to agent i.

We believe that the phenomenon of extinction reversal is of fundamental importance for understanding survival of heterogeneous agents. While our goal is only to construct an example of extinction reversal, we think that this phenomenon is quite universal and should occur in many models of competitive behavior and has important implications for studying long-run competition. Suppose, for example, that we are analyzing competition of firms in a market and we want to know which firms will dominate in the long run. Typically, economic models addressing this question assume that there are only two firms in the economy (see, e.g., Kogan, Ross, Wang and Westerfield (2006)). Our results indicate that the question cannot be answered without studying what other firms are present in the market, because their presence may lead to extinction reversal.

The main ingredients responsible for the occurrence of extinction reversal in our model are: (1) the fact that the allocation is Pareto-efficient and (2) the fact that the agents are maximizing utility only from wealth (consumption) at the terminal time T.* Under these conditions, there are two effects that drive the equilibrium terminal wealth of an agent: the wealth allocation effect and the welfare weight effect. The wealth allocation effect determines how the agent allocates wealth across the states of the world. This is an agent-specific effect and it does not depend on the characteristics of other agents in the economy. It is determined by the agent's characteristics such as risk attitude, beliefs and production technology. By contrast, the welfare weight effect is of a completely different nature. Since the allocation is Pareto-efficient, there exist welfare weights such that the equilibrium allocation maximizes a linear combination of all the agents' utilities, multiplied with these weights. However, these weights are determined endogenously in equilibrium and depend on the characteristics of all the agents in the economy, as well as on the terminal horizon T. In particular, as the horizon T increases, the rate at which the welfare weight of an agent in the economy is changing with T will depend on the characteristics of other agents. Therefore, the rates of relative extinction of two agents will also depend on the parameters of other agents in the economy.

In our example we consider an economy populated by two agents who only differ in their risk aversions and show that the agent whose preference are closest to that of a log agent (an agent with constant relative risk aversion equal to one) dominates in the long run and the other agent gets extinct. Now, suppose that we add a third agent to the economy whose preferences are closer to the log agent than those of the two other agents. Then, in the long run, only the third agent survives and owns the whole economy. As the horizon increases, the impact of the third agent on the total welfare grows and, consequently, the long run behavior of equilibrium welfare weights is

*The assumption of no intermediate consumption is often used in the literature on survival. See e.g. Amir, Evstigneev, Hens, and Schenk-Hoppe (2005). It allows us to concentrate on the effects of capital accumulation. We get our results sending T to infinity, which should be interpreted through the extinction rates, explicitly derived in our paper: if the extinction of i relative to j takes place at rate ρ, it means that, effectively, if the horizon T is larger than ρ^{-1}, the market share of agent i is negligible relative to that of agent j.

strongly affected by the third agent. We calculate the exact effect of the third agent on the rates of extinction of the other two agents and determine the set of parameters for which extinction reversal occurs. More precisely, the risk aversions of the two agents must be on the different sides of one, and the risk aversion of the third agent must be sufficiently close to one, at a distance not more than the geometric mean of the risk aversions of the other two agents.

We now discuss the related literature. It has become a conventional wisdom that, with no intermediate consumption, the agent with logarithmic utility of terminal wealth will have the highest wealth growth rate and thus will eventually dominate in the long run. See, e.g., Rubinstein (1991), Blume and Easley (1992), Kogan, Ross, Wang and Westerfield (2006). One consequence of our results is that the agent whose preferences are the closest to logarithmic dominates in the long run.[†]

Most of the existing literature on survival studies the case when the agents maximize utility of intertemporal consumption. Blume and Easley (2009b) is an excellent survey of the existing results. The main general feature of all equilibrium survival models with intermediate consumption is that the long run survival of each agent is characterized by a single number, the “survival index,” that depends only the agent’s characteristics. The first result of this kind has been discovered by Sandroni (2000). He showed that, when markets are complete, the economy is bounded and all agents have identical discount factors, then an agent’s survival index is given by the entropy of his subjective probability measure relative to the true probability measure. In particular, only the agents with the smallest entropy survive. Blume and Easley (2006) extended the results of Sandroni, allowing for very general learning mechanisms and heterogeneous discount rates. They showed that the survival result holds in any Pareto optimal allocation in any bounded economy, and thus for any complete markets equilibrium. For bounded complete markets economies there is a survival index that determines which traders survive, which traders vanish, and also determines the rates of relative extinction of different traders.

[†]We only consider the case when all agents have constant relative risk aversion (CRRA) utilities. The results can be extended to the case of general utilities (see Cvitanić and Malamud (2009a)), but the analysis gets substantially more technical and we confine ourselves to the CRRA case for the reader’s convenience.

This index depends only on the agent's characteristics and exogenous state variables, namely, the agent's discount factor, the actual stochastic process of the states and the agent's beliefs about this stochastic process. Interestingly enough, for these economies, attitudes toward risk do not matter for survival. They also show that the market selects the traders who learn the true process over those who do not learn the truth, the Bayesians with the true value in the support of their prior over comparable non-Bayesians, and among Bayesians according to the dimension of the support of their prior (assuming that the true value is in the support). However, Blume and Easley (2006) provide only necessary conditions for survival. Blume and Easley (2009a) consider a special case of the Blume and Easley (2006) model when the agents have heterogeneous beliefs, but do not learn. They provide a detailed study of the case when there are multiple agents with equal maximal survival indices and show which of them do, indeed, survive in the long run. Yan (2009) shows that risk preferences matter when the economy is unbounded. Similarly to Blume and Easley (2006), he shows that survival and extinction of each agent is characterized by a survival index but this index depends on the agent's risk aversion. The conclusion is that in the models with intermediate consumption and complete markets, extinction reversal cannot occur because relative extinction is *independent of the characteristics of other agents*. The reason is that, with intermediate consumption and complete markets, the welfare weight effect is absent: the horizon is infinite from the beginning and the welfare weights are fixed. Thus, long run wealth dynamics are determined by two allocation effects: allocation across states and allocation across time periods. These effects are determined by the agent's beliefs, discount rate and elasticity of intertemporal substitution, and are not affected by the parameters of other agents in the economy.

Little is known about survival when markets are incomplete. Sandroni (2005) shows that in markets with an incomplete set of Arrow securities an agent's survival and relative extinction depend only on the agent's characteristics (namely, the entropy of his beliefs relative to the true probability measure) and thus the result of Blume and Easley still holds. However, for multi-period incomplete markets, the situation is more complicated (see Blume and Easley (2009b) for a survey of existing results). The reason is that equilibrium allocations are not Pareto efficient and welfare weights are stochastic and evolve through time. Thus, the welfare weight effect is present in incomplete markets with intermediate consumption: in the long run, the dominant agents will have higher weights and will impact relative extinction of other agents and may potentially lead to extinction reversal. We leave this interesting problem for future research.

Blume and Easley (2002) consider a different survival problem. They address the famous conjecture of Milton Friedman that only the profit-maximizing firms should survive in the long run. They consider a deterministic dynamic equilibrium model with firms that use their retained earnings for investment and show that, indeed, only the profit-maximizing firms survive. However, surprisingly, the long-run state of the evolutionary process is inefficient. In contrast to our paper, Blume and Easley (2002) do not study relative extinction. In the first part of the paper, they do not allow capitalists (the firm owners) to trade in the financial markets and show that the profit maximizing capitalists always survive. However, it is not clear from their analysis which of the capitalists will disappear faster and how quickly (i.e., at which rate) this extinction will happen. In the second part of the paper, they study the case when capitalists can reallocate capital between firms by trading in financial markets. However, they do not derive general results for survival/extinction, but present several examples indicating that even capitalists with incorrect beliefs may dominate in the long run. However, their examples only deal with the case of two capitalists and it is thus unclear how the presence of more than two capitalists in the market affects their relative extinction.

The literature discussed above addresses exclusively the "preference-based" approach to survival. Alternatively, one could study survival when investors follow rules exhibiting particular behaviors. Amir, Evstigneev, Hens, and Schenk-Hoppe (2005) and Evstigneev, Hens, and Schenk-Hoppe (2006) consider general one-period markets and study whether there are simple portfolio rules that survive in the long run, or are evolutionarily stable, when the market is populated by other simple portfolio rules. A simple rule is one for which the fraction of wealth invested in a given asset is independent of the current asset prices. In this research, either all the winnings are invested (which corresponds to the no intermediate consumption case of our model), or equivalently the traders are assumed to invest an equal fraction of their winnings and consumption rates are the same for all traders. Thus, market selection depends only on portfolio rules and not on the agent's wealth allocation needs across states and time periods. Amir, Evstigneev, Hens, and Schenk-Hoppe (2005) show that the trader who allocates his wealth across assets according to their conditional expected relative payoffs dominates in the long run. Evstigneev, Hens, and Schenk-Hoppe (2006) show that the expected relative payoffs rule is evolutionarily stable using notions of stability from evolutionary game theory. Amir, Evstigneev, Hens and Xu (2009) extend this analysis in a game-theoretic framework and allow for general, adaptive strategies (portfolio rules), distributing their wealth between assets, depending on the exogenous states of the world and the observed history of the game.

The paper is organized as follows. Section 2 introduces the model and defines the equilibrium allocation. Section 3 contains the main results – the expressions for the rates of extinction and conditions for extinction reversal. Section 4 concludes and the proofs are delegated to Appendix.

2 Setup and notation

2.1 The model

We consider a standard setting analogous to that of Wang (1996), except there is only a terminal dividend. The economy has a finite horizon and evolves in continuous time. Uncertainty is described by a one-dimensional, standard Brownian motion $B_t,\ t \in [0\,,\,T]$ on a complete probability space $(\Omega, \mathcal{F}_T, P)$, where $\mathcal{F}$ is the augmented filtration generated by B_t. There is a single share of a risky asset in the economy, the stock, which pays a terminal dividend payment

$$D \;= D_T = \; e^{\rho T \,+\, \sigma B_T}.$$

We also assume that a risk-free asset with instantaneous constant rate r is available in zero net supply.[‡] The price of the stock at time t is denoted by

[‡]The existence of an instantaneously risk-free security is a common assumption in equilibrium literature (see, e.g., Wang (1996)). When time is discrete, it is commonly assumed that a one-period risk free bond is available for trading. In the continuous time limit,

S_t. The instantaneous drift and volatility of the stock price S_t are denoted by μ_t and σ_t respectively,

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dB_t.$$

There are K competitive agents in the economy, who behave rationally, and are heterogeneous in risk preferences. Agent k is initially endowed with ψ_k shares of the stock at time zero, and we have

$$\sum_k \psi_k = 1.$$

Agent k chooses portfolio strategy π_{kt}, the portfolio weight in the risky asset, so as to maximize the CRRA expected utility

$$E\left[\frac{W_{kT}^{1-\gamma_k}}{1-\gamma_k}\right]$$

of final wealth W_{kT}, where the wealth W_{kt} evolves as

$$dW_{kt} = W_{kt}(rdt + \pi_{kt}(S_t^{-1}dS_t - rdt)).$$

Remark 2.1. All the results of the paper can be directly extended to the case of agents having heterogeneous beliefs on the expected return rate of the endowment (as, e.g., in Sandroni (2000), Blume and Easley (2006) and Yan (2009)), and to utility functions generalizing CRRA utilities. However, the analysis for the latter becomes much more technical, and the details are available from the authors upon request.

2.2 The equilibrium

Definition 2.1. *We say that the market is in equilibrium if the agents behave optimally and both the risky asset market and the risk-free market clear.*

It is well known that the above financial market is complete if the volatility process σ_t of the stock price is almost everywhere strictly positive. See Duffie

the one-period bonds become instantaneously risk-free. The assumption of constant r is introduced only for simplicity of exposition.

(2001, Chapter 6).[§] When the market is complete, there exists a unique stochastic discount factor (SDF) $M = M_T$ such that the stock price is given by

$$S_t \;=\; e^{r(t-T)}\,\frac{E_t[MD]}{E_t[M]}.$$

The quantity $M/E_t[M]$ is often referred to as the Arrow-Debreu state price density. Namely, when the state space is discrete, with state s occurring with probability p_s, we can construct Arrow securities that pay 1 in state s and zero otherwise. By market completeness, Arrow securities can be replicated by trading in the existing securities (stock and bonds). Consequently, holding an asset paying the dividend d_s in state s is equivalent to holding a bundle of Arrow securities, and the price P of this asset coincides with the price of the bundle,

$$P \;=\; \sum_s d_s\,\Pi_s$$

where Π_s is the price of the Arrow security for state s. The vector $M = (m_s) = (\Pi_s/p_s)$ is referred to as the state price density. It allows us to write the pricing formula as

$$P \;=\; E[M\,D].$$

This expression directly extends to the case of a continuous state space. In that case Arrow-Debreu state prices are not defined, but the state price density is well defined. Since the markets are complete, the agent can trade either in the directly available assets or in Arrow securities. The market equilibrium in which agents trade in Arrow securities is called an Arrow-Debreu equilibrium. When the state space is discrete (continuous), an Arrow-Debreu equilibrium is characterized by the equilibrium state prices (the state price density).

Consequently, in our model, because of market completeness, the equilibrium allocation is Pareto-efficient and can be characterized as an Arrow-Debreu equilibrium.[¶] See, e.g. Duffie (1986), Wang (1996). Because the

[§]It is not difficult to see from standard arguments that the stock price is strictly monotone increasing in W_t and hence the volatility is strictly positive. In fact, it is possible to show (see, Cvitanić and Malamud (2009b)) that the stock price volatility is always larger than the dividend volatility.

[¶]It should be mentioned that the market in our setting is dynamically complete, so that the Arrow securities can be replicated by dynamic rather that static portfolios.

endowments are co-linear (all agents hold shares of the same single stock), the equilibrium is in fact unique, up to a multiplicative factor, and unique if we fix the risk-free rate. See, e.g., Dana (1995), Dana (2001).[‖]

Due to market completeness, an agent's optimal wealth can be calculated directly through the Arrow-Debreu state prices (the state price density). Given agent k's financial wealth $W_{k\,0}$ at time zero, the agent can attain any contingent claim W_T satisfying the budget constraint

$$E[M_T\,W_T] \;\leq\; W_{k\,0}.$$

Thus, agent k is maximizing

$$E[u_k(W_{k\,T})] \;-\; \lambda_k\,(E[M_T\,W_{k\,T}] \;-\; W_{k\,0})$$

where λ_k is the Lagrange multiplier and $u_k(x) \;=\; x^{1-\gamma_k}/(1-\gamma_k)$ is the agent's utility. The first order condition immediately implies that the optimal terminal wealth is of the form

$$W_{kT} \;=\; (\lambda_k\,M)^{-b_k}$$

where

$$b_k \;=\; \gamma_k^{-1}$$

is the relative risk tolerance of agent k, and λ_k is determined via the *budget constraint*

$$E[(\lambda_k M)^{-b_k}\,M] \;=\; W_{k0} \;=\; \psi_k\,S_0 \;=\; \psi_k\,E[DM].$$

We formalize our findings in

Proposition 2.1. *The equilibrium allocation is given by*

$$W_{kT} \;=\; \frac{\psi_k\,E[DM]}{E[M^{1-b_k}]}\,M^{-b_k}$$

[‖]Since the endowment is neither bounded away from zero nor from infinity, some additional care is needed to show the existence of equilibrium. See, e.g., Dana (2001) and Malamud (2008a).

and the equilibrium SDF M solves the equation

$$\sum_k \frac{\psi_k\, E[DM]}{E[M^{1-b_k}]}\, M^{-b_k} \;=\; D. \tag{1}$$

3 Relative extinction

Definition 3.2. *We say that a function $f(T)$ converges to zero almost at rate $\rho > 0$ as $T \to \infty$ if for any $\epsilon > 0$ there exist constants $K_1(\epsilon),\ K_2(\epsilon)$ such that*

$$K_1(\epsilon)\, e^{-(\rho+\epsilon)\,T} \;\le\; f(T) \;\le\; K_2(\epsilon)\, e^{-(\rho-\epsilon)\,T}$$

for all $T \ge 0$.

We will also need the following

Definition 3.3. *We say that agent i experiences extinction relative to agent j if*

$$\lim_{T\to\infty} \frac{W_{iT}}{W_{jT}} \;=\; 0$$

almost surely.

We start with the following result, the first part of which confirms the intuition of Rubinstein (1991).

Theorem 3.1. *Suppose that there exists a unique agent 0 such that* **

$$(1-\gamma_0)^2 \;=\; \min_k\, (1-\gamma_k)^2.$$

Then,

$$\frac{W_{kT}}{W_{0T}} \;\to\; 0$$

for all $k \neq 0$ and the convergence happens almost at rate $-s_k$ with

$$s_k \;\stackrel{def}{=}\; b_k\,(-(1-\gamma_k)^2 + (1-\gamma_0)^2) \;<\; 0\,.$$

**The assumption is true for generic values of risk aversion. On the other hand, if there are two agents equally distant from the log, they will both survive and share the economy in the long run.

We call s_k the *survival index* of agent k. Theorem 3.1 shows that this index determines how quickly an agent gets extinct relative to agent 0 closest to the log agent, and that the rate of extinction is influenced by two effects: (i) the distance of the agent's risk aversion from risk aversion 1 relative to that distance for the single surviving agent 0; (ii) the agent's risk tolerance. The first effect is intuitive: if the agent is far away from the log agent in terms of risk aversion, the long-run growth rate of his wealth is small and, consequently, his survival index is also small. The appearance of risk tolerance b_k multiplying the "distance" $-(1-\gamma_k)^2 + (1-\gamma_0)^2$ of agent k from agent 0 is a new and unexpected effect. Intuitively, small risk tolerance (large risk aversion) diminishes the effect of the "distance" from the log because the agent takes safe positions in the stock, investing more into the bonds, so that his wealth fluctuations are smaller. In contrast, when his risk tolerance is large, the agent takes risky positions, his wealth fluctuates more, and thus the effect of the distance factor gets magnified.

Writing down

$$\frac{W_{iT}}{W_{jT}} = \frac{W_{iT}/W_{0T}}{W_{jT}/W_{0T}},$$

Theorem 3.1 implies

Corollary 3.1. *An agent i experiences extinction relative to agent j if and only if*

$$s_i < s_j$$

and W_{iT}/W_{jT} converges to zero almost at rate $s_j - s_i$.

The results of Theorem 3.1 and Corollary 3.1 are crucial for understanding the nature of extinction and survival. We are particularly interested in the phenomenon of extinction reversal. Namely, we say that an extinction reversal occurs if the order in which the agents i and j get extinct in the multiple agent economy is reverse to that in the economy in which only agents i and j are present.

As we mention in the introduction, in all the existing literature on survival in heterogeneous economies with intermediate consumption, the rate at which agent i vanishes in the long run relative to agent j depends only on the preferences of the corresponding agent and the properties of the aggregate endowment. Theorem 3.1 shows that this is not true in our model: the rate depends on the risk aversion γ_0 of the surviving agent.

We will now formulate necessary and sufficient conditions for extinction reversal. A direct calculation shows that

$$s_j - s_i = \frac{\gamma_i - \gamma_j}{\gamma_i \gamma_j} \left(\gamma_i \gamma_j - 1 + (1 - \gamma_0)^2 \right). \tag{2}$$

In particular, if $\gamma_i > \gamma_j > 1$, we get $s_j > s_i$. Similarly, if $\gamma_i < \gamma_j < 1$, we have

$$(1 - \gamma_0)^2 < (1 - \gamma_i)^2 < 1 - \gamma_i < 1 - \gamma_i \gamma_j$$

and we again get $s_j > s_i$. Thus, if the risk aversions of agents i and j are on the same side of 1, the relative extinction does not depend on the presence of agent 0 – the agent further away from the risk aversion of 1 gets extinct relative to the other agent. If $\gamma_i > 1 > \gamma_j$ but $\gamma_i \gamma_j > 1$ then we also get $s_j > s_i$. Suppose now that $\gamma_j > 1 > \gamma_i$ but $\gamma_i \gamma_j < 1$. Then, in an economy populated by agents i and j only, agent j will dominate in the long run if and only if $\gamma_j - 1 < 1 - \gamma_i$. In this case, a direct calculation shows that

$$(\gamma_j - 1)^2 < 1 - \gamma_i \gamma_j.$$

Therefore, introducing a new surviving agent in the economy will not lead to extinction reversal. Indeed, the inequality $(1-\gamma_0)^2 < (1-\gamma_j)^2 < 1-\gamma_i\gamma_j$ and (2) imply that $s_j > s_i$. Thus, we can only get extinction reversal if $\gamma_i - 1 > 1 - \gamma_j > 0$ and the quantity in the brackets on the right hand side of (2) is negative. We formalize our findings in

Corollary 3.2. *Let $\gamma_j < 1$ and suppose that $\gamma_i > 2 - \gamma_j$ and $\gamma_i \gamma_j < 1$. Then agent i experiences extinction relative to agent j in the economy populated only by agents i and j. On the other hand, if we also have*

$$\gamma_0 \in \left(1 - \sqrt{1 - \gamma_i \gamma_j}\,,\, 1 + \sqrt{1 - \gamma_i \gamma_j}\right) \tag{3}$$

then extinction reversal occurs in any economy in which all three agents i, j and 0 are present, irrespective of the risk aversions of other agents. These conditions are both necessary and sufficient.

As we explain in the introduction, there are two effects determining the size of the agent's wealth: the wealth allocation effect and the welfare weight effect. Agent k's wealth is given by

$$W_{kT} \;=\; M^{-b_k}\,\frac{\psi_k\,E[DM]}{E[M^{1-b_k}]}.$$

The long run behavior of the part M^{-b_k} is driven by the wealth allocation effect. More precisely, the risk tolerance b_k determines how the agent allocates his wealth across states. If the risk aversion is high, risk tolerance is low and the agent hedges against "bad" states with low consumption and puts a lot of weight on them. That is, effectively, the agent bets on the realization of bad states. By contrast, if the agent's risk aversion is low, he is willing to take on more risks, buys more stock and, effectively, bets on the realization of "good" states with high consumption. The log agent is exactly "in the middle" between the very risk averse and very risk tolerant agents and bets on "average" (not too good and not too bad states). This wealth allocation rule leads to optimal wealth growth. The constant factor $z_k = \frac{\psi_k\,E[DM]}{E[M^{1-b_k}]}$ comes from the welfare weight effect. Its long run behavior depends in a nontrivial way on the characteristics of other agents in the economy. When the horizon T is large, the impact of the surviving agent 0 of this factor becomes large and affects the extinction rate of agent k. In the framework of Corollary 3.2 this effect is particularly strong. The reason is as follows. When only two agents i and j are present in the economy, agent j dominates because his risk aversion is closer to 1 and, for this reason, he has a higher growth rate of his wealth. In particular, the state price density M_T behaves asymptotically as $D_T^{-\gamma_j}$, i.e., as if agent i were not present in the economy. However, in the presence of agent 0, state price density M_T behaves asymptotically as $D_T^{-\gamma_0}$ and is therefore more volatile (because $\gamma_0 > \gamma_j$). Agent j, being less risk averse than γ_0, invests a larger portion of his wealth into stock, which also makes his wealth very volatile and makes him dominated by agent 0 faster than agent i. The latter is more risk averse than agent 0, invests less into stock and has therefore a less volatile wealth process.

Put differently, Condition (3) requires that γ_0 be very close to 1. If agent 0 is very close to being logarithmic, survival indices of other agents in the economy will become small and he will quickly own the whole economy. His presence will then lead to dramatic changes in prices, and make the strategy of more risk averse agent i "better" relative to that of less risk averse agent j. In effect, a greater part of the utility welfare of agent j than that of agent i has been taken over by agent 0. On the other hand, in infinite horizon models with intermediate consumption and complete markets, the analog of the constant factor z_k does not depend on T and so the welfare weight effect is absent, and extinction reversal cannot occur.

4 Conclusions

In a complete market model with CRRA agents maximizing utility from terminal wealth only, we show that the agent whose relative risk aversion is closest to the log agent is the only surviving agent, asymptotically, as the horizon tends to infinity. We find the rate at which other agents get extinct and show that this rate depends on the agent's risk tolerance and on how far away the agent is from the log agent relative to how far away is the surviving agent from the log agent. Because the surviving agent's risk aversion affects how fast other agents get extinct, it is possible to have extinction reversal: if a new agent is added to an economy of two agents, in this new economy the agent who originally gets extinct may survive longer than the originally surviving agent. This phenomenon cannot happen in complete markets with agents maximizing utility from consumption over infinite horizon. On the other hand, it would be of considerable interest to study similar phenomena in incomplete markets.

Appendix

A Proofs

When risk aversion is homogeneous across agents, the equilibrium SDF is explicitly determined by $D^{-\gamma}/E[D^{-\gamma}]$. However, when risk aversion is heterogeneous, the SDF is the solution to highly non-linear equation (1), and no explicit solution is possible, except for some very special values of risk aversion; see, for example, Wang (1996). In the lemma below, we establish bounds on the equilibrium SDF.

Lemma A.1. *Let $\Gamma \geq 1$ be such that $\Gamma\, b_i > 1$ for all i and $\gamma \leq 1$ be such that $\gamma b_i \leq 1$ for all i. Then,*

$$\left(\sum_i D^{-\gamma_i/\gamma}\big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i/\gamma}\right)^{\gamma} \leq M \leq \left(\sum_i D^{-\gamma_i/\Gamma}\big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i/\Gamma}\right)^{\Gamma}. \quad (4)$$

Quantity $D^{-\gamma_i}\big(\psi_i\, E[DM]/E[M^{1-b_i}]\big)^{\gamma_i}$ can be viewed as the "individual" SDF in an economy populated only by agent i. It is known that, when risk aversion is heterogeneous, the equilibrium SDF can be represented as a generalized weighted Hölder average of the "individual" SDFs (see, e.g., Malamud (2008a), Malamud (2008b), Jouini and Napp (2008), Shefrin (2005)). Lemma A.1 shows that M can be estimated from both below and above by Hölder averages with different exponents γ and Γ.

Proof of Lemma A.1: Let

$$z_i = \psi_i\, E[DM]/E[M^{1-b_i}]\,.$$

Then, the equilibrium equation is

$$\sum_i z_i\, M^{-b_i} = D.$$

Suppose that

$$M \; > \; \left(\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma} \right)^{\Gamma} .$$

Then,

$$\begin{aligned} \sum_i z_i \, M^{-b_i} \, D^{-1} \; &< \; \sum_i z_i \left(\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma} \right)^{-\Gamma/\gamma_i} \\ &= \; \sum_i \left(\frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}} \right)^{\Gamma/\gamma_i} . \end{aligned} \quad (5)$$

Since $\Gamma > \gamma_i$ for all i, we get

$$\left(\frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}} \right)^{\Gamma/\gamma_i} \; < \; \frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}}$$

and therefore

$$\sum_i z_i \, M^{-b_i} \, D^{-1} \; < \; \sum_i \frac{D^{-\gamma_i/\Gamma} z_i^{\gamma_i/\Gamma}}{\sum_i D^{-\gamma_i/\Gamma} \, z_i^{\gamma_i/\Gamma}} \; = \; 1$$

which is a contradiction. The estimate from below follows by the same argument. ∎

The weights $\left(\psi_i \, E[DM]/E[M^{1-b_i}]\right)^{\gamma_i}$ are not directly helpful for getting useful bounds for M. The following useful lemma allows us to obtain uniform bounds for these weights. It has a very clear economic meaning: the maximal utility of an agent is larger than the utility from simply consuming its endowment, and is smaller than the utility from consuming the aggregate endowment of the economy.

Lemma A.2. *Let M be the equilibrium SDF. If $\gamma_i < 1$ then*

$$1 \; \le \; \frac{E[DM]^{1-\gamma_i} \, E[M^{1-b_i}]^{\gamma_i}}{E[D^{1-\gamma_i}]} \; \le \; \psi_i^{\gamma_i - 1}.$$

If $\gamma_i > 1$ then

$$\psi_i^{\gamma_i - 1} \le \; \frac{E[DM]^{1-\gamma_i} \, E[M^{1-b_i}]^{\gamma_i}}{E[D^{1-\gamma_i}]} \; \le \; 1.$$

Proof: The utility of agent i's optimal wealth is given by

$$\frac{1}{1-\gamma_i}\,E[W_{iT}^{1-\gamma_i}] \;=\; \frac{1}{1-\gamma_i}\psi_i^{1-\gamma_i}\,\left(\frac{E[DM]}{E[M^{1-b_i}]}\right)^{1-\gamma_i}\,E[M^{1-b_i}]$$
$$= \;\frac{1}{1-\gamma_i}\psi_i^{1-\gamma_i}E[DM]^{1-\gamma_i}\,E[M^{1-b_i}]^{\gamma_i}. \quad (6)$$

The utility from just consuming its endowment (the terminal dividend of its initial portfolio) is

$$\frac{1}{1-\gamma_i}\,E[(\psi_i D)^{1-\gamma_i}] \;=\; \frac{1}{1-\gamma_i}\psi_i^{1-\gamma_i}\,E[D^{1-\gamma_i}].$$

Furthermore, by definition, in equilibrium we must have $W_{iT} \;\le\; D$ and therefore

$$\frac{1}{1-\gamma_i}\,E[(\psi_i D)^{1-\gamma_i}] \;\le\; \frac{1}{1-\gamma_i}\,E[W_{iT}^{1-\gamma_i}] \;\le\; \frac{1}{1-\gamma_i}\,E[D^{1-\gamma_i}]\,.$$

Multiplying both sides by $1-\gamma_i$ and using (6), we get the result. ■

Lemmas A.1 and A.2 together will allow us to obtain good bounds on the ratio W_{iT}/W_{jT}.

Lemma A.3. *There exist constants* $K_1,\;K_2>0$ *such that*

$$K_1\sum_i\frac{E[DM]}{E[D^{1-\gamma_i}]}\,D^{-\gamma_i} \;\le\; M \;\le\; K_2\sum_i\frac{E[DM]}{E[D^{1-\gamma_i}]}\,D^{-\gamma_i}.$$

Proof: By Lemma A.1,

$$\frac{1}{n}\sum_i\,D^{-\gamma_i}\big(\psi_i\,E[DM]/E[M^{1-b_i}]\big)^{\gamma_i} \le\; M$$
$$\le\; n\,\sum_i\,D^{-\gamma_i}\big(\psi_i\,E[DM]/E[M^{1-b_i}]\big)^{\gamma_i}. \quad (7)$$

By Lemma A.2,

$$K_1\frac{E[DM]}{E[D^{1-\gamma_i}]} \;\le\; \big(\psi_i\,E[DM]/E[M^{1-b_i}]\big)^{\gamma_i} \;\le\; K_2\frac{E[DM]}{E[D^{1-\gamma_i}]}$$

for some $K_1,\ K_2 > 0$. The proof is complete. ■

We will also need the following

Lemma A.4. *For any $\alpha > 0$ and any $x_i > 0$,*

$$\min\{n^{\alpha-1}, 1\} \sum_i x_i^\alpha \ \le\ \left(\sum_i x_i\right)^\alpha \ \le\ \max\{n^{\alpha-1}, 1\} \sum_i x_i^\alpha.$$

Proof of Theorem 3.1. Let first $\gamma_i > \gamma_0$. By Lemma A.2,

$$\hat{K}_2 \ \le\ \frac{W_{iT}/W_{0T}}{\frac{E[D^{1-\gamma_0}]^{b_0}}{E[D^{1-\gamma_i}]^{b_i}} E[DM]^{b_i-b_0} M^{b_0-b_i}} \ \le\ \hat{K}_1.$$

for some $\hat{K}_1,\ \hat{K}_2 > 0$. Combining Lemmas A.3 and A.4, we get that

$$\tilde{K}_2 \ \le\ \frac{M^{b_0-b_i}}{\sum_k \frac{E[DM]^{b_0-b_i}\, D^{(b_i-b_0)\gamma_k}}{E[D^{1-\gamma_k}]^{b_0-b_i}}} \ \le\ \tilde{K}_1.$$

and therefore W_{iT}/W_{0T} converges to zero almost at the same rate as

$$\begin{aligned}\sum_k & \frac{E[D^{1-\gamma_0}]^{b_0}\, D^{(b_i-b_0)\gamma_k}}{E[D^{1-\gamma_i}]^{b_i} E[D^{1-\gamma_k}]^{b_0-b_i}} \\ &= \sum_k e^{\frac{1}{2}\sigma^2 T \left((1-\gamma_0)^2 b_0 - (1-\gamma_i)^2 b_i - (1-\gamma_k)^2 (b_0-b_i) + 2\,\sigma^{-1}\,\gamma_k (b_i-b_0)\, B_T/T\right)}. \qquad (8)\end{aligned}$$

Since, by assumption, $b_i = \gamma_i^{-1} < \gamma_0^{-1} = b_0$,

$$\begin{aligned}\max_k & \{(1-\gamma_0)^2 b_0 - (1-\gamma_i)^2 b_i - (1-\gamma_k)^2 (b_0 - b_i)\} \\ &= (1-\gamma_0)^2 b_0 - (1-\gamma_i)^2 b_i - (1-\gamma_0)^2 (b_0 - b_i) \ =\ s_i \qquad (9)\end{aligned}$$

and the claim follows since, by the strong law of large numbers for the Brownian motion, $B_T/T \to 0$ almost surely.

Finally, if $\gamma_i < \gamma_0$, we can repeat the same argument and show that $W_{iT}/W_{0T} \to 0$ almost at rate $-s_i$, which is what had to be proved.

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